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Some Classes of Banach Spaces with Normal Structure

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Abstract: **In this paper, we prove that if a Banach space $(X, \|.\|)$ has unconditional Schauder basis $\{e_n\}$ with unconditional basis constant c and has normal structure for weakly compact convex sets, then the equivalent renorming $(X, |.|_\beta)$, (where $|x|_\beta = max\{\|x\|, \beta q(x)\}, \beta > 0, q$ is a seminorm on *X*) has normal structure for weakly compact convex sets under a suitable condition. Also, we give an alternate proof for normal structure of a *k*-uniform rotundity (*k*-UR) Banach space.**

Keywords: **Normal structure, fixed point property, unconditional basis, *k*-uniform rotundity (*k*-UR) Banach space.**

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1. Introduction

Let K be a nonempty weakly compact convex subset of a Banach space X. A mapping $T: K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$. We say, K has the fixed point property (fpp) if every nonexpansive map on K has a fixed point in K. Also, we say X has the fixed point property (fpp) if every nonempty weakly compact convex subset of X has the fixed point property. In 1965, Kirk [10] proved that every Banach space with normal structure has the fpp. The idea of normal structure was introduced by Brodskii and Milman in 1948 [3] to study the existence of common fixed point of isometrics. It is known that every compact convex subset of a Banach space has normal structure [7], hence every finite dimensional Banach space has the fpp. Browder and Gohde [7] proved the existence of fixed points for nonexpansive mappings in a uniformly convex Banach space without using normal structure. Note that, every uniformly convex Banach space has normal structure and hence every Hilbert space has normal structure [7]. In 1976, Karlovitz [8] showed that normal structure is not a needful condition for the fpp of a Banach space by extending Kirk's work. Subsequently, Maurey [4,13] proved c_0 with the sup norm has the fpp though it does not have normal structure [7]. However, we can not drop the weakly compact assumption in Kirk's theorem [7]. Since then, different types of geometric conditions introduced by various authors which guaranteed the fpp for a Banch Space (one can refer [1, 2, 4, 5, 7, 9, 11, 12]) although the problem: Does every reflexive Banach space have the fpp?- remains open.

More, specifically the problem: Does every equivalent renorming of $(l^P, \|.\|_p), 1 < p < \infty,$ have the fpp?- also remains open. Note that $(l^P, \|.\|_p), 1 < p < \infty,$ is uniformly convex and hence has the fpp [7]. Further, every equivalent renorming of $(l^P, \|.\|_p), 1 < p < \infty,$ is reflexive. In [5], it was proved that the equivalent renorming $(l^2, |.|_\beta),$ (where $|x|_\beta = max\{\|x\|_2, \beta q(x)\}, \beta > 0, q$ is a seminorm on l^2) has normal structure if for each sequence $\{y_k\}$ with $\|y_k\|_2 \leq 1, supp(y_k)$ is finite and $max(supp(y_k)) < min(supp(y_{k+1})),$ for all $k,$ there exist $n_0 \in \mathbb{N}$ and $\alpha > 0$ such that $q(y_k) \leqslant \alpha < 1/\beta,$ for all $k \geq n_0.$. In this paper, we generalize the above result for a Banach space with unconditional Schauder basis and from this we give examples of some equivalent renormings of $(l^P, \|.\|_p), 1 < p < \infty,$ with normal structure. In fact, here we prove the following: Let$(X, \|.\|)$be a Banach space with unconditional Schauder basis $\{e_n\}$ and c be the unconditional basis constant. Suppose $(X, \|.\|)$ has normal structure for weakly compact convex sets. Let $|x|_\beta = max\{\|x\|, \beta q(x)\}$ (where $\beta > 0$ and q is a seminorm on X) be an equivalent renorming of $(X, \|.\|)$. Then $(X, |.|_\beta)$ has normal structure for weakly compact convex sets, if for each sequence $\{y_k\}$ with $\|y_k\| \leq c, supp(y_k)$ is finite and $max(supp(y_k)) < min(supp(y_{k+1})),$ for all $k,$ there exist $n_0 \in \mathbb{N}$ and $\alpha > 0$ such that $q(y_k) \leqslant \alpha < 1/\beta,$ for all $k \geq n_0.$

Sullivan [1979] introduced the notion of k-uniform rotundity $(k \in \mathbb{N})$ for a Banach Space X which is a generalization of uniform convexity and showed that these Banach spaces have normal structure [15]. In this article, we give an alternate proof for the normal structure of a k-uniform rotundity (k-UR) Banach space.

2. Main Results

Let X be a Banach space and $A \subset X$. We use the following notations:

$$r_x(A) = \sup\{\|x - y\|: y \in A\},$$

$$diam(A) = \text{ diameter of } A,$$

$$co(A) = \text{convex hull of } A,$$

$$\overline{co}(A) = \text{convex hull closure of } A.$$

Definition 2.1. [3,7] *A nonempty convex subset K of a Banach space X is said to have normal structure if for every nonempty closed bounded convex subset C of K with $diam(C) > 0$, there exist $x \in C$ such that $r_x(C) < diam(C)$.*

If every convex subset of a Banach space X has normal structure, then we say that X has normal structure. Also, we say X has normal structure for weakly compact convex sets if every weakly compact convex subset of X has normal structure.

In 1965, Kirk proved the following:

Theorem 2.2. [10] *Let K be a weakly compact convex subset of a Banach space X. If K has normal structure, then K has the fpp.*

The following theorem gives a characterization for normal structure.

Theorem 2.3. [7] *A nonempty bounded convex subset K of a Banach space X has normal structure if and only if it does not contain a non constant sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} dist\,(x_{n+1}, co\{x_1, x_2, \dots, x_n\}) = diam\{x_1, x_2, \dots, x_n, \dots\}.$$

Corollary 2.4. [7] *A Banach space X has normal structure for weakly compact convex sets if and only if it does not contain a sequence $\{x_n\}$ such that*

$$x_n \to^{w} 0$$

$$\|x_n\| \to 1$$

$$diam(\{x_n\}) = 1.$$

Let X be a Banach space and $\{e_n\}$ be a Schauder basis of X. Let $x \in X$. Then there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{n=1}^{\infty} \alpha_n e_n$. The support of x with respect to $\{e_n\}$ is

$$supp(x) = \{n \in \mathbb{N} : \alpha_n \neq 0\}.$$

Definition 2.5. [14] *A Schauder basis $\{e_n\}$ of a Banach space X is said to be unconditional if for each $x \in X$, the expansion $x = \sum_{n=1}^{\infty} \alpha_n e_n$ for x in terms of the basis is unconditionally convergent, that is $\sum_{n=1}^{\infty} \alpha_{\pi(n)} e_{\pi(n)}$ converges for any permutation π of $\mathbb{N}$.*

Theorem 2.6. [14] *Suppose $\{e_n\}$ is an unconditional basis for a Banach space X. For each $A \subset \mathbb{N}$, define the linear operator $P_A : X \to X$, as $P_A(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n\in A} \alpha_n e_n$. Then P_A, for all $A \subset \mathbb{N}$, is a* *bounded linear operator and* $\sup\{\|P_A\|: A \subset \mathbb{N}\}$ *is finite. The constant* $c = \sup\{\|P_A\|: A \subset \mathbb{N}\}$ *is called unconditional basis constant for the basis* $\{e_n\}$.

Theorem 2.7. [14] *Suppose* $\{e_n\}$ *is an unconditional basis for a Banach space* X *and* c *is the unconditional basis constant. Then* c *is the smallest real number such that*

$$\left\|\sum_{n\in A} \alpha_n e_n\right\| \leq c \left\|\sum_{n=1}^{\infty} \alpha_n e_n\right\|$$

whenever $x = \sum_{n=1}^{\infty} \alpha_n e_n \in X$ *and* $A \subset \mathbb{N}$.

Lemma 2.8. *Let* X *be a Banach space with unconditional Schauder basis* $\{e_n\}$ *and* c *be the unconditional basis constant. Suppose* $\{x_n\}$ *is a sequence in* X *which converges weakly to 0. Then there is a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and a sequence* $\{u_k\}$ *such that*

(i) $\|x_{n_k} - u_k\| \to 0$

(ii) $\|u_k\| \leq c\|x_{n_k}\|$, *for all* k.

(iii) There exist positive integers $N_0, N_1, \ldots, N_k, \ldots$ *such that* $N_{k-1} < N_k, k = 1,2, \ldots.$ *and* $supp(u_k) \subset (N_{k-1}, N_k]$, *where* $(N_{k-1}, N_k] = \{j \in \mathbb{N}: N_{k-1} < j \leq N_k\}$.

Proof: Suppose $\{P_n\}$ is the sequence of natural projections with respect to the Schauder basis $\{e_n\}$ and $\{\epsilon_k\}_{k\geq 0}$ is a sequence of positive real numbers converging to 0.

Then $\|P_n(x_1) - x_1\| \to 0$ as $n \to \infty$. Therefore there exists a $N_0 \in \mathbb{N}$ such that $\|P_{N_0}(x_1) - x_1\| < \epsilon_0.$ Let $n_0 = 1$.

Now since $\{x_n\}$ converges weakly to 0, $\|P_{N_0}(x_n)\| \to 0$ as $n \to \infty$. So there exists a $n_1 > 1$ such that $\|P_{N_0}(x_{n_1})\| < \epsilon_1.$

Again $\|P_n(x_{n_1}) - x_{n_1}\| \to 0$ as $n \to \infty$. Therefore there exists a $N_1 > N_0$ such that $\|P_{N_1}(x_{n_1}) - x_{n_1}\| < \epsilon_1.$

Now since $\|P_{N_1}(x_n)\| \to 0$ as $n \to \infty$. So there exists a $n_2 > n_1$ such that $\|P_{N_1}(x_{n_2})\| < \epsilon_2.$

Continuing this process we get $n_1 < \cdots < n_k < \cdots$ and $N_0 < N_1 < \cdots < N_k < \cdots$ such that $\|P_{N_k}(x_{n_k}) - x_{n_k}\| < \epsilon_k$ and $\|P_{N_{k-1}}(x_{n_k})\| < \epsilon_k$.

Let $u_k = (P_{N_k} - P_{N_{k-1}})(x_{n_k})$. Then $\|x_{n_k} - u_k\| \leq \|P_{N_k}(x_{n_k}) - x_{n_k}\| + \|P_{N_{k-1}}(x_{n_k})\| < 2\epsilon_k$.

Therefore $\|x_{n_k} - u_k\| \to 0$ and $supp(u_k) \subset (N_{k-1}, N_k], N_{k-1} < N_k, k = 1,2, \ldots.$

Since $\{e_n\}$ is an unconditional basis with unconditional basis constant c, by Theorem 2.7., we have

$$\|u_k\| = \|(P_{N_k} - P_{N_{k-1}})(x_{n_k})\| \leq c\|x_{n_k}\|.$$

Theorem 2.9. *Let* $(X, \|.\|)$ *be a Banach space with unconditional Schauder basis* $\{e_n\}$ *and* c *be the unconditional basis constant. Suppose* $(X, \|.\|)$ *has normal structure for weakly compact convex sets. Let*

$|x|_\beta = max\{\|x\|, \beta q(x)\}$ *(where* $\beta > 0$ *and* q *is a seminorm on* X*) be an equivalent renorming of* $(X, \|.\|)$. *Then* $(X, |.|_\beta)$ *has normal structure for weakly compact convex sets, if for each sequence* $\{y_k\}$ *with* $\|y_k\| \le c, supp(y_k)$ *is finite and* $max(supp(y_k)) < min(supp(y_{k+1}))$, *for all* k, *there exist* $n_0 \in \mathbb{N}, \alpha > 0$ *such that* $q(y_k) \le \alpha < 1/\beta, \forall k \ge n_0$.

Proof: Suppose $(X, |.|_\beta)$ does not have normal structure for weakly compact convex sets. Then by Corollary 2.4., there exists a sequence $\{x_n\}$ such that

$$x_n \to^w 0$$

$$|x_n|_\beta \to 1$$

$$diam_\beta(\{x_n\}) = 1.$$

Then by Lemma 2.8., there exists a subsequence $\{x_{n_k}\}$ and a sequence $\{u_k\}$ such that $\|x_{n_k} - u_k\| \to 0, supp(u_k) \subset (N_{k-1}, N_k]$, where $N_{k-1} < N_k, k = 1,2, \ldots.$ and $\|u_k\| \le c\|x_{n_k}\|$, for all k. Since $|.|_\beta$ is an equivalent renorming of $\|.\|, |x_{n_k} - u_k|_\beta \to 0$.

Now since $diam_\beta(\{x_n\}) = diam_\beta(\overline{co}\{x_n\}) = 1$, $x_n \to^w 0, 0 \in \overline{co}\{x_n\}$ and $|x_n|_\beta \le 1$.

Therefore, $\|u_k\| \le c\|x_{n_k}\| \le c|x_{n_k}|_\beta \le c$.

Let f be a bounded linear functional on X. Since $x_{n_k} \to^w 0, f(x_{n_k}) \to 0$.

Therefore, $|f(u_k)| \le |f(u_k - x_{n_k})| + |f(x_{n_k})| \le \|f\||x_{n_k} - u_k|_\beta + |f(x_{n_k})|$ implies $f(u_k) \to 0$.

So $u_k \to^w 0$.

Now $|x_{n_k}|_\beta - |u_k - x_{n_k}|_\beta \le |u_k|_\beta \le |x_{n_k}|_\beta + |u_k - x_{n_k}|_\beta$ implies that $|u_k|_\beta \to 1$.

From the given condition of the theorem, there exist $n_0 \in \mathbb{N}, \alpha > 0$ such that $q(u_k) \le \alpha < 1/\beta, \forall k \ge n_0$. Then $\beta q(u_k) \le \alpha\beta < 1, \forall k \ge n_0$. Therefore $|u_k|_\beta = max\{\|u_k\|, \beta q(u_k)\} \to 1$ gives $\|u_k\| \to 1$.

Thus $(X, \|.\|)$ contains a sequence $\{u_k\}$ such that

$$u_k \to^w 0$$

$$\|u_k\| \to 1$$

$$diam(\{u_k\}) = 1.$$

Then by Corollary 2.4., $(X, \|.\|)$ does not have normal structure for weakly compact convex sets.

Corollary 2.10. [5] *The equivalent renorming of* l^2 *of the form* $|x|_\beta = max\{\|x\|_2, \beta q(x)\}$ *(where* $\beta > 0$ *and* q *is a seminorm on* X*) has normal structure, if for each sequence* $\{y_k\}$ *with* $\|y_k\| \le 1, supp(y_k)$ *is finite and* $max(supp(y_k)) < min(supp(y_{k+1}))$, *for all* k, *there exist* $n_0 \in \mathbb{N}, \alpha > 0$ *such that* $q(y_k) \le \alpha <$

$1/\beta, \forall k \geq n_0.$

Proof: The standard basis $\{e_n\}$ of $(l^2, \|.\|_2)$, where $e_n = (x(1), x(2), \ldots), x(n) = 1, x(i) = 0, \forall n \neq i$, is an unconditional basis [14] with unconditional constant 1. Also since $(l^2, \|.\|_2)$, is reflexive, every closed bounded set is weakly compact. Therefore the conclusion follows from Theorem 2.9.

Example 2.11. *Consider* $l^p (1 < p < \infty)$ *with the norm* $\|x\| = \left\{\|x\|_p, \beta \sup_n \left\{\frac{|x(1)|+\cdots+|x(n)|}{|a_n|}\right\}\right\}$, *where* $\{a_n\}$ *is a sequence of non zero real numbers and there exist* $n_0 \in \mathbb{N}, \alpha > 0$ *such that* $|n/a_n| \leq \alpha < 1/\beta, \forall k \geq n_0.$ *Then* $(l^p, \|.\|)$ *has normal structure.*

Proof: It is known that the standard basis $\{e_n\}$ of $(l^p, \|.\|_p)$, where $e_n = (x(1), x(2), \ldots), x(n) = 1, x(i) = 0, \forall n \neq i$, is an unconditional basis [14] with unconditional constant 1. Since $(l^p, \|.\|_p)$ is uniformly convex, it is reflexive and has normal structure. Now since $|n/a_n| \leq \alpha < 1/\beta, \forall k \geq n_0, \{n/a_n\}$ is bounded. Let $|n/a_n| \leq M, \forall n \in \mathbb{N}$, for some $M > 0$.

Therefore $\|x\|_p \leq \|x\| = \left\{\|x\|_p, \beta \sup_n \left\{\frac{|x(1)|+\cdots+|x(n)|}{|a_n|}\right\}\right\} \leq M_1 \|x\|_p$, for all $x \in l^p$, for some $M_1 > 1$. Hence $\|.\|$ is an equivalent renorming of $\|.\|_p$.

Let $\{y_k\}$ be a sequence in l^p such that $\|y_k\|_p \leq 1, supp(y_k)$ is finite and $max(supp(y_k)) < min(supp(y_{k+1}))$, for all k. Then there exist positive integers $N_1 < N_2 < \cdots < N_k$ such that $supp(y_k) \subset (N_{k-1}, N_k]$.

Let $q(x) = \sup_n \left\{\frac{|x(1)|+\cdots+|x(n)|}{|a_n|}\right\}, \forall x \in l^p$.

Now, $q(y_k) = \sup \left\{\frac{|y_k(1)|}{|a_1|}, \cdots, \frac{|y_k(1)|+\cdots+|y_k(N_k)|}{|a_{N_k}|}, \cdots, \frac{|y_k(1)|+\cdots+|y_k(N_{k+1}-1)|}{|a_{N_{k+1}-1}|}, \cdots\right\}$

$\leq \sup \left\{\frac{|N_K|}{|a_{N_k}|}, \cdots, \frac{|N_{K+1}-1|}{|a_{N_{k+1}-1}|}, \cdots, \right\} \leq \alpha$, for large k.

Thus from Theorem 2.9., $(l^p, \|.\|)$ has normal structure.

In 1979, Sullivan [15] introduced the notion of *k*-uniformly rotund (*k*-UR) Banach spaces and proved these spaces have normal structure. Here we provide a proof for normal structure. For this, we need the following in sequel.

Definition 2.12. [15] *A Banach space* X *is said to be k-uniformly rotund (k-UR) if every* $\epsilon > 0$,

$$\delta_X^k(\epsilon) = \inf\left\{1 - \frac{\left\|\sum_{i=1}^{k+1} x_i\right\|}{k+1} : \|x_i\| \leq 1, i = 1, \ldots, (k+1), V(x_1, \cdots, x_{k+1}) \geq \epsilon\right\} > 0,$$

where, $V(x_1,\dots,x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_i \in X^*, \|f_i\| \le 1, i = 1,\dots,k\right\}.$

It is easy to see that 1-UR Banach space is uniformly convex. It is known every *k*-UR is reflexive and has normal structure [15].

For $x_1, x_2, \dots, x_m (m > 1)$ in a Banach space define,

$$F(x_1, x_2, \dots, x_m) = dist(x_1, co\{x_2, \dots, x_m\}) dist(x_2, co\{x_3, \dots, x_m\}) \cdots \|x_{m-1} - x_m\|.$$

Theorem 2.13. *Let* X *be a Banach space and* $m(> 1)$ *be a positive integer. Suppose for a given* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that for all* $x_1, x_2, \dots, x_m \in X$ *with* $\|x_i\| \le 1 (i = 1, \dots, m)$,

$$F(x_1, x_2, \dots, x_m) < \epsilon \tag{1.1}$$

$$\text{whenever } \left\|\frac{x_1 + \cdots + x_m}{m}\right\| > 1 - \delta.$$

Then X *has normal structure.*

Proof: Suppose X does not normal structure. Then by Theorem 2.3., there exists a sequence $\{x_n\}$ such that $lim_{n\to\infty} dist\,(x_{n+1}, co\{x_1, x_2, \dots, x_n\}) = diam\{x_1, x_2, \dots, x_n, \dots\}$. Assume that $diam\{x_1, x_2, \dots, x_n, \dots\} = 1$. Choose $0 < p < 1$ and let $\epsilon = (1 - p)$.

Then there exists a $\delta > 0$ such that (1.1) holds.

Let $s_1^L = x_{L+1} - x_{L+m+1}, s_2^L = x_{L+2} - x_{L+m+1}, \cdots, s_m^L = x_{L+m} - x_{L+m+1}$, for all $L \in \mathbb{N}$.

Now for large L,

$$\left\|\frac{s_1^L + \cdots + s_m^L}{m}\right\| > 1 - \delta.$$

For $1 \le i \le m$,

$$dist\big(s_{i+1}^L, co\{s_1^L, \dots, s_i^L\}\big) = dist(x_{L+i+1}, co\{x_{L+1}, \dots, x_{L+i}\}) > 1 - p, \text{ for large } L.$$

Now,

$$1 - p < dist(s_m^L, co\{s_1^L, \dots, s_{m-1}^L\}) dist(s_{m-1}^L, co\{s_1^L, \dots, s_{m-2}^L\}) \cdots \|s_2^L - s_1^L\|$$

$$= F(s_m^L, \dots, s_1^L) < \epsilon$$

which is a contradiction for large L.

Theorem 2.14. [6] *Suppose* $x_1, x_2, \dots, x_n (n > 1)$ *are vectors in Banach space with* $\|x_i\| \le 1$, *for all* $i = 1,2, \dots, n$. *Then* $F(x_1, x_2, \dots, x_n) \le V(x_1, x_2, \dots, x_n)$, *where* $V(x_1, x_2, \dots, x_n)$ *is defined as above.*

Theorem 2.15. *Every k-UR Banach space has normal structure.*

Proof: We show that X satisfies Theorem 2.13., for $m = (k+1)$. Let $\epsilon > 0$.

Then since X is k-UR, $\delta_X^k(\epsilon) > 0$, where δ_X^k is as in the definition of k-UR.

Let $\delta = \delta_X^k(\epsilon)$. Let $x_1, x_2, \dots, x_{k+1} \in X$ with $\|x_i\| \leq 1$ and

$$\left\|\frac{x_1 + \cdots + x_{k+1}}{k+1}\right\| > 1 - \delta.$$

Now suppose $F(x_1, x_2, \dots, x_{k+1}) \geq \epsilon$. So by Theorem 2.14., $V(x_1, x_2, \dots, x_{k+1}) \geq \epsilon$.

Then from the definition of k-UR, we have

$$\left\|\frac{x_1 + \cdots + x_{k+1}}{k+1}\right\| \leq 1 - \delta_X^k(\epsilon) = 1 - \delta.$$

This is a contradiction.

3. Conclusion

Here in the Theorem 2.9., we prove that $(X, |.|_\beta)$ has normal structure for weakly compact convex sets under the assumption $(X, \|.\|)$ having an unconditional Schauder basis but we do not know the theorem is true or not for a Banach space $(X, \|.\|)$ with Schauder basis (not necessarily unconditional).

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